

# Strengthening Hardness Results to 3-Connected Planar Graphs

Giordano Da Lozzo<sup>1</sup> and Ignaz Rutter<sup>2</sup>

<sup>1</sup> Department of Engineering, Roma Tre University, Italy

<sup>2</sup> Karlsruhe Institute of Technology, Germany

**Abstract.** In this paper we extend some classical NP-hardness results from the class of 2-connected planar graphs to subclasses of 3-connected planar graphs. The reduction are partly based on a new graph augmentation, which may be of independent interest.

## 1 Introduction

When proving NP-hardness results for graph drawing problems on planar graphs with variable embedding it is often necessary to restrict the embedding choices of a construction. In this case it is convenient to reduce from problems that are NP-complete for subclasses of 3-connected planar graphs, which have an almost unique combinatorial embedding. While there exist many hardness results for 2-connected planar graphs, only few results are known for the 3-connected case.

In this paper we strengthen some classical NP-hardness results to this setting. We show NP-hardness for MAXIMUM INDEPENDENT SET (MIS) for 3-connected cubic planar graphs and planar triangulations, 3-Coloring for 3-connected bounded-degree planar graphs, and STEINER TREE for 3-connected cubic planar graphs.

The reductions for MIS and 3-COLORING are based on a new graph augmentation technique (Lemma 1) to transform 2-connected planar graphs with bounded degree into 3-connected planar graphs with bounded degree by “subdividing” each edge at most once, which may be of independent interest.

We have recently used the hardness of MIS for 3-connected planar graphs for showing hardness of an embedding problem that asks to maximize the number of facial cycles that are contained in a given set  $\mathcal{C}$  [6]. Similarly, our result on Steiner trees can be used to simplify and extend hardness results for embedding problems (e.g., [3, Theorem 8], [2, Theorem 7], and [1, Theorem 17]).

## 2 Preliminaries

We assume familiarity with basic concepts of graph drawing and planarity (see, e.g., [4]). For the definition of the SPQR-tree of a biconnected graph and the concepts of *skeleton*  $\text{skel}(\mu)$  and *pertinent graph*  $\text{pert}(\mu)$  of a node  $\mu$  of an SPQR-tree, and that of *virtual edge* of a skeleton, and *expansion graph* of a virtual edge we refer the reader to [5]. For the definition of *canonical ordering* we refer the reader to [9]. For convenience we also provide definitions in Appendix A.

Let  $G = (V, E)$  be a plane graph with two designated edges  $e', e'' \in E$  incident to the same face. A *graph augmentation* on the pair  $\langle e' = (u, v), e'' = (w, z) \rangle$  turns  $G$  into a new planar graph  $G'$  by replacing edges  $e'$  and  $e''$  with a connected planar graph  $G_A$  containing four vertices each of which is identified with one of  $\{u, v, w, z\}$  in such a way that  $G_A$  is planar. We say that a graph augmentation is *k-good* if it does not increase the number of  $k$ -cuts in the graph. An *H-split* on the pair  $\langle e', e'' \rangle$  is a graph augmentation that turns  $G$  into a new plane graph  $G'$  by subdividing edges  $e'$  and  $e''$  with a dummy vertex  $v'$  and  $v''$ , respectively, and by adding edge  $(v', v'')$ . Clearly, an *H-split* is 2-good.

**NP-hard Problems.** An independent set in a graph  $G = (V, E)$  is a subset  $V' \subseteq V$  of pairwise non-adjacent vertices. The problem MIS asks for a maximum size independent set. A 3-coloring of a graph  $G = (V, E)$  is an assignment  $c: V \rightarrow \{1, 2, 3\}$  such that for every edge  $(u, v) \in E$  it is  $c(u) \neq c(v)$ . The problem 3-COLORING asks whether a given graph admits a 3-coloring. Let  $(G, T)$  be a pair where  $G = (V, E)$  is a graph and  $T \subseteq V$  is a set of *terminals*. A Steiner tree is a subtree of  $G$  that contains all vertices in  $T$ . The problem STEINER TREE asks for an instance  $(G, T)$  for a smallest Steiner tree of  $G$ , where the size is measured in terms of the number of edges.

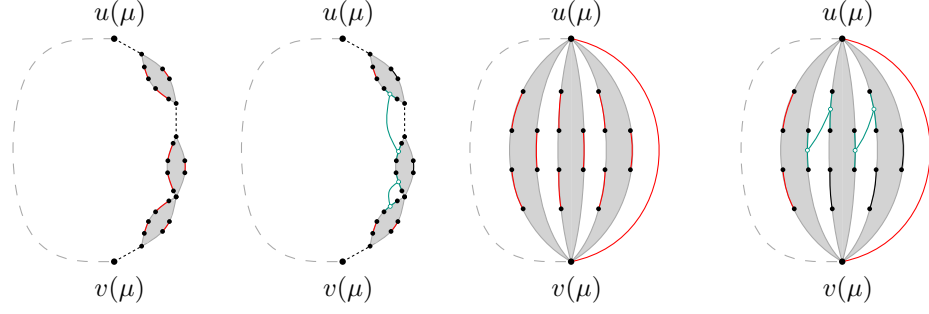
### 3 Bounded-degree Augmentation

In this section we give an algorithm (Lemma 1) to augment a 2-connected planar graph  $G$  with minimum degree  $\delta(G) \geq 3$  and maximum degree  $\Delta(G)$  to a 3-connected planar graph  $G'$  with  $\delta(G') = \delta(G)$  and  $\Delta(G') = \Delta(G)$  by applying *H-splits* to disjoint pairs of edges of  $G$ .

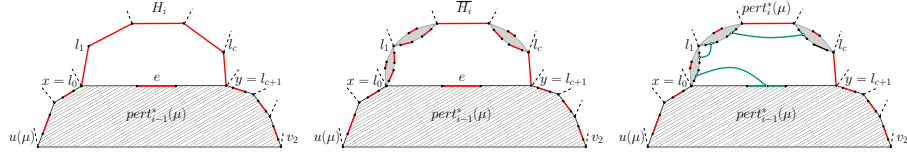
A planar embedding of  $\text{pert}(\mu)$  is *regular* if the parent edge is incident to the outer face. Let  $\mathcal{E}_\mu$  be a planar embedding of  $\text{pert}(\mu)$  and let  $e$  be an edge of  $\text{pert}(\mu)$  that is incident to the outer face after removing the parent edge. Embedding  $\mathcal{E}_\mu$  is *e-externally 3-connectible* if either 1.  $\mu$  is a Q-node or 2. the graph obtained from  $\text{pert}(\mu)$  by performing an *H-split* on  $e$  and the parent edge is a subdivision of a 3-connected planar graph whose only degree-2 vertices, if any, are the poles of  $\mu$ . Also, we say that  $\mu$  is *e-externally 3-connectible* (or, simply, *externally 3-connectible*) if  $\text{pert}(\mu)$  admits an *e-externally 3-connectible* embedding, for some edge  $e$  of  $\text{pert}(\mu)$ .

Let  $\mu$  be a node of the SPQR-tree  $\mathcal{T}$  of  $G$ , let  $e$  be an edge of  $\text{pert}(\mu)$ , and let  $\text{pert}^*(\mu)$  be a graph obtained by applying *H-splits* on distinct pairs of edges in  $\text{pert}(\mu)$ . We say that  $e$  is a *free edge* if  $e \in E(\text{pert}(\mu)) \setminus E(\text{pert}^*(\mu))$ , that is, edge  $e$  has not been used in any *H-split*.

Consider a non-Q-node  $\mu$ . Let  $\text{pert}^*(\mu)$  be a graph obtained from  $\text{pert}(\mu)$  via a set of edge-disjoint *H-splits*, let  $L_\mu = [e_1, e_2]$  and  $R_\mu = [e_3]$  be two lists of free edges in  $E(\text{pert}^*(\mu))$ , and let  $\mathcal{E}_\mu^*$  be a regular embedding of  $\text{pert}^*(\mu)$ . We say that  $\mathcal{E}_\mu^*$  is *extendible* if  $L_\mu$  and  $R_\mu$  are incident to different faces of  $\mathcal{E}_\mu^*$  incident to the parent edge and  $\mathcal{E}_\mu^*$  is *e-externally 3-connectible* with  $e \in L_\mu$ .



**Fig. 1:** Augmentation of an  $S$ -node (a–b) and of a  $P$ -node (c–d) via  $H$ -splits. Free edges are red, while the vertices and edges created by  $H$ -splits are green.



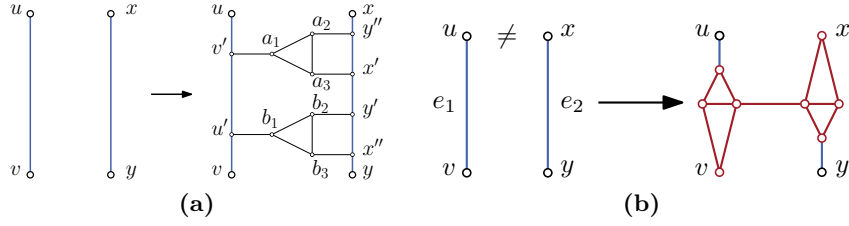
**Fig. 2:** Augmentation of an  $R$ -node via  $H$ -splits for the next part of the canonical ordering. Free edges are red, vertices and edges created by  $H$ -splits are green.

The proof is based on inductively constructing an extendible embedding of each node  $\mu$  of the SPQR-tree assuming that extendible embeddings exist for the children of  $\mu$ . For  $S$ -nodes and  $P$ -nodes the construction is straightforward; see Fig. 7. Note that after the augmentation there are two free edges on one side of the embedding and one on the other side, which satisfies our invariant.

For an  $R$ -node  $\mu$  we construct an extendible embedding by processing the vertices of  $\text{skel}(\mu)$  according to a canonical ordering as illustrated in Fig. 8. The construction allows us to satisfy our invariant by embedding the augmentations of the children of  $\mu$  such that they contribute free edges to the outer face in each step of the canonical ordering. Special care has to be taken if the edge from which the canonical ordering starts does not correspond to a  $Q$ -node.

Since the described augmentation only uses  $H$ -splits, it does not increase the minimum and maximum degree beyond 3. We have the following main result.

**Lemma 1.** *Let  $G = (V, E)$  be a 2-connected planar graph with minimum degree  $\delta(G) \geq 3$  and maximum degree  $\Delta(G)$ . There exist disjoint pairs  $(e'_1, e''_1), \dots, (e'_k, e''_k)$  of edges in  $E$  such that performing the  $H$ -splits  $\langle e'_1, e''_1 \rangle, \dots, \langle e'_k, e''_k \rangle$  yields a 3-connected planar graph  $G'$  with  $\delta(G') = \delta(G)$  and  $\Delta(G') = \Delta(G)$ .*



**Fig. 3:** (a) Gadget for the proofs of Lemma 2. (b) Gadget  $A_1((e_1, e_2))$  used in the hardness proof of 3-Coloring. Before the substitution the endpoints  $u, v$  and  $x, y$  of  $e_1$  and  $e_2$ , respectively, are exchanged so that  $u \neq x$  holds.

## 4 Hardness Results Based on 2-Good Augmentations

In this section we give examples on how to exploit Lemma 1 to extend NP-hardness results from the class of 2-connected planar graphs with minimum degree 3 and bounded maximum degree to that of 3-connected planar graphs with bounded maximum degree. The general line is as follows. Given an NP-hard decision problem  $\mathcal{P}$  which takes as input a planar graph  $G$  and, possibly, a parameter  $k$ , one just needs to define a graph augmentation which (i) is 2-good and (ii) replaces a pair of edges  $\langle e', e'' \rangle$  of  $G$  with a gadget  $A(\langle e', e'' \rangle)$  of polynomial size to obtain a graph  $G'$  such that  $(G, k)$  is a **yes** instance for problem  $\mathcal{P}$  if and only if  $(G', k')$  is a **yes** instance for problem  $\mathcal{P}$ , where  $k' = f(k)$  and  $f$  is a computable polynomial function.

**Maximum Independent Set.** For a graph  $G$  we denote by  $\alpha(G)$  the size of a largest independent set in  $G$ .

**Lemma 2.** *Let  $G$  be a 2-connected cubic planar graph and let  $e$  and  $e'$  be two edges in  $E(G)$  incident to the same face of a planar embedding of  $G$ . Let  $G'$  be the 2-connected cubic planar graph obtained from  $G$  by applying the graph augmentation illustrated in Fig. 3a to  $e$  and  $e'$ . Then  $\alpha(G') = \alpha(G) + 5$ .*

*Proof.* Let  $I$  be an independent set in  $G$ . We construct an independent set  $I'$  of  $G'$  as follows. We start with  $I$ . If  $I$  does not contain  $u$ , we add  $v'$  to it; otherwise it does not contain  $v$ , and we can add  $u'$ . Similarly, if  $I$  does not contain  $x$ , we add  $y', y''$  as well as  $a_3$  and  $b_3$ ; otherwise it does not contain  $y$ , and we add  $x', x''$  as well as  $a_2$  and  $b_2$ . Clearly  $I'$  is an independent set of size  $|I| + 5$ , showing that  $\alpha(G') \geq \alpha(G) + 5$ .

Conversely, assume that  $I'$  is an independent set of  $G'$ . First assume that  $a_1 \in I'$ . Note that  $I'$  cannot contain both  $x'$  and  $y''$ . It follows that we can replace  $a_1$  by either  $a_2$  or by  $a_3$  to obtain an independent set of the same size. An analogous argument applies to  $b_1$ . We can hence assume without loss of generality that neither  $a_1$  nor  $b_1$  is contained in  $I'$ . Now assume that  $I'$  contains both  $u$  and  $v$ . It then follows that  $u'$  and  $v'$  are not in  $I'$ , and hence  $(I' \setminus \{v\}) \cup \{u'\}$  is an

independent set (since  $b_1 \notin I'$ ) of the same size. We can hence also assume that  $I'$  contains at most one vertex in  $\{u, v\}$ .

Next assume that  $\{x, y\} \subseteq I'$ . Then  $I'$  contains at most one vertex in  $\{x', y', x'', y''\}$ , at most one vertex in  $\{a_2, a_3\}$ , and at most one vertex in  $\{b_2, b_3\}$ . Then  $I' \setminus (\{x, y\} \cup \{x', y', x'', y''\} \cup \{a_2, a_3\} \cup \{b_2, b_3\}) \cup \{x, x', x'', a_2, b_2\}$  is a larger independent set containing only one vertex in  $\{x, y\}$ . Hence, we can also assume without loss of generality that  $I'$  contains at most one vertex in  $\{x, y\}$ .

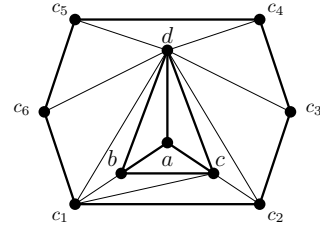
It follows from the above that, after suitably transforming  $I'$ , the set  $I' \cap V(G)$  is an independent set in  $G$ . Now observe that  $I'$  can contain at most one vertex from each of the sets  $\{u', v'\}$ ,  $\{a_1, a_2, a_3\}$ ,  $\{b_1, b_2, b_3\}$ ,  $\{x', y''\}$ , and  $\{x'', y'\}$ . It follows that  $|I' \cap V(G)| \geq |I'| - 5$ . In particular, this implies  $\alpha(G) \geq \alpha(G') - 5$ , or, equivalently,  $\alpha(G') \leq \alpha(G) + 5$ .  $\square$

By applying Lemma 2 to the distinct pairs of edges of a biconnected cubic planar graph (for which MIS is NP-complete [12]) determined by Lemma 1 we obtain the following.

**Theorem 1.** *MIS is NP-complete for 3-connected cubic planar graphs.*

**3-Coloring.** Let  $\mathcal{C}(G)$  denote the number of cut vertices of a graph  $G$ . Given a pair of edges  $\langle e_1 = (u, v), e_2 = (x, y) \rangle$  of a planar graph  $G$ , we define gadget  $A_1(\langle e_1, e_2 \rangle)$  as the graph illustrated in Fig. 3b, where we assume that  $u \neq x$ . Observe that gadget  $A_1(\langle e_1, e_2 \rangle)$  is 2-good. We first prove an auxiliary lemma.

**Lemma 3.** *The 3-COLORING problem is NP-complete for 2-connected planar graphs with minimum degree 4 and maximum degree 7.*



**Fig. 4:** Gadget  $\Phi_6$  for placement inside a face of size 6.

We can now exploit Lemma 3 and Lemma 1 and the fact that gadget  $A_1$  is 2-good to obtain the following theorem.

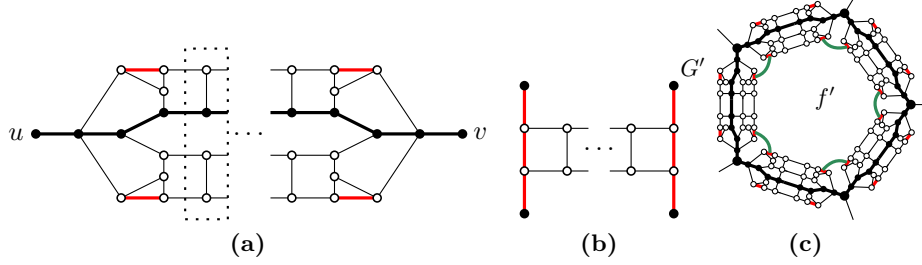
**Theorem 2.** *The 3-COLORING problem is NP-complete for 3-connected planar graphs with minimum degree 4 and maximum degree 7.*

## 5 Other Hardness Results for 3-Connected Planar Graphs

In this section we present further strengthenings of hardness results to 3-connected planar graphs. The difference to the previous section is that the proofs do not make use of the graph augmentation technique of Lemma 1.

By replacing one non-triangular face of size  $l$  of a plane graph by the gadget  $\Phi_l$ , see Fig. 4, we can prove the following.

**Lemma 4.** *Let  $G$  be a 2-connected plane graph and  $f_{\geq 4}(G)$  be the number of faces of  $G$  whose size is larger than 3. There exists a 2-connected plane graph  $G'$  such that (i)  $\alpha(G') = \alpha(G) + 1$  and (ii)  $f_{\geq 4}(G') = f_{\geq 4}(G) - 1$ .*



**Fig. 5:** Hardness of STEINER TREE for 3-connected cubic graphs.

It is known that MIS is NP-complete for 2-connected planar graphs [12]. Iteratively applying the construction from Lemma 4 yields the following.

**Theorem 3.** *MIS is NP-complete for planar triangulations.*

Next, we show that STEINER TREE is NP-complete for 3-connected cubic graphs. We first show that STEINER TREE is NP-complete for planar graphs of maximum degree 3 by a reduction from STEINER TREE in planar graphs, which is known to be NP-complete[10].

**Lemma 5.** *STEINER TREE is NP-complete for biconnected planar graphs of maximum degree 3.*

**Theorem 4.** *STEINER TREE is NP-complete for 3-connected cubic planar graphs.*

*Proof.* We reduce from STEINER TREE in biconnected planar graphs of maximum degree 3. Let  $(G, T)$  be such an instance and fix an arbitrary planar embedding of  $G$ . We call a *chain* a maximal path whose internal vertices have degree 2. After subdividing each edge with eight subdivision vertices, we can assume that each chain has length at least 9. We now replace each chain of length  $\ell$ , whose endpoints are degree-3 vertices  $u$  and  $v$ , by a copy of the gadget from Fig. 5a, where the size is chosen such we can identify the original chain with the bold path in the gadget. Denote by  $G'$  the resulting graph together with its induced embedding from  $G$ . For the terminal set we choose  $T' = T$ . Observe that  $G$  is a subgraph of  $G'$  (bold paths in the gadgets). Moreover, since the bold paths in the gadgets are shortest paths between the two endpoints, it follows that for any pair of vertices in  $G$  there is a shortest path in  $G'$  that uses only vertices in  $G$ . Hence  $(G, T)$  and  $(G', T')$  are equivalent instances of STEINER TREE. Observe further that  $G'$  is cubic. It remains to make it 3-connected.

To complete the construction, for each face  $f$  of  $G$ , we traverse its boundary and for any two consecutive chains with endpoints  $u, v$  and  $v, w$  in such a face, we perform in  $G'$  an augmentation operation on the two red edges  $e, e'$  that are closest to  $v$  and that are incident to the corresponding face  $f'$  in  $G'$  using a ladder gadget of sufficient length (Fig. 5b) so that the ladder gadget cannot be used as

a shortcut. Finally, observe that the augmentation operations ensure that the final graph is 3-connected and moreover, they do not change the lengths of the bold paths; see Fig. 5c for an illustration of the whole reduction. The resulting instance  $(G', T')$  is hence still equivalent to the original instance  $(G, T)$ . The reduction can clearly be carried out in polynomial time.  $\square$

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## A Preliminaries and Definitions

### A.1 Drawings and Embeddings

A *planar drawing*  $\Gamma$  of a graph maps vertices to points in the plane and edges to internally disjoint curves. Drawing  $\Gamma$  partitions the plane into topologically connected regions, called *faces*. The bounded faces are *internal* and the unbounded face is the *outer face*. A planar drawing determines a circular ordering of the edges incident to each vertex. Two planar drawings of a connected planar graph are *equivalent* if they determine the same orderings and have the same outer face. A *combinatorial embedding* is an equivalence class of planar drawings.

### A.2 Connectivity and SPQR-trees

A graph  $G$  is *connected* if there is a path between any two vertices. A *cutvertex* is a vertex whose removal disconnects the graph. A *separating pair* is a pair of vertices  $\{u, v\}$  whose removal disconnects the graph. A connected graph is *2-connected* if it does not have a cutvertex and a 2-connected graph is *3-connected* if it does not have a separating pair. A 2-connected plane graph  $G$  is *internally 3-connected* if  $G$  can be extended to a 3-connected planar graph by adding a vertex in the outer face and joining it to all the vertices incident to the outer face.

We consider  $uv$ -graphs with two special *pole* vertices  $u$  and  $v$ , which can be constructed in a fashion very similar to series-parallel graphs. Namely, an edge  $(u, v)$  is an  $uv$ -graph with poles  $u$  and  $v$ . Now let  $G_i$  be an  $uv$ -graph with poles  $u_i, v_i$  for  $i = 1, \dots, k$  and let  $H$  be a planar graph with two designated vertices  $u$  and  $v$  and  $k + 1$  edges  $uv, e_1, \dots, e_k$ . We call  $H$  the *skeleton* of the composition and its edges are called *virtual edges*; the edge  $uv$  is the *parent edge* and  $u$  and  $v$  are the poles of the skeleton  $H$ . To compose the  $G_i$  into an  $uv$ -graph with poles  $u$  and  $v$ , we remove the edge  $uv$  and replace each  $e_i$  by  $G_i$  for  $i = 1, \dots, k$  by removing  $e_i$  and identifying the poles of  $G_i$  with the endpoints of  $e_i$ . In fact, we only allow three types of compositions: in a *series composition* the skeleton  $H$  is a cycle of length  $k + 1$ , in a *parallel composition*  $H$  consists of two vertices connected by  $k + 1$  parallel edge, and in a *rigid composition*  $H$  is 3-connected.

It is known that for every 2-connected graph  $G$  with an edge  $uv$  the graph  $G - st$  is an  $uv$ -graph with poles  $u$  and  $v$ . Much in the same way as series-parallel graphs, the  $uv$ -graph  $G \setminus uv$  gives rise to a (de-)composition tree  $\mathcal{T}$  describing how it can be obtained from single edges. The nodes of  $\mathcal{T}$  corresponding to edges, series, parallel, and rigid compositions of the graph are *Q*-, *S*-, *P*-, and *R*-nodes, respectively. To obtain a composition tree for  $G$ , we add an additional root *Q*-node representing the edge  $uv$ . To fully describe the composition, we associate with each node  $\mu$  its skeleton denoted by  $\text{skel}(\mu)$ . For a node  $\mu$  of  $\mathcal{T}$ , the *pertinent graph*  $\text{pert}(\mu)$  is the subgraph represented by the subtree with root  $\mu$ . Similarly, for a virtual edge  $\varepsilon$  of a skeleton  $\text{skel}(\mu)$ , the *expansion graph* of  $\varepsilon$ , denoted by  $\text{exp}(\varepsilon)$  is the pertinent graph  $\text{pert}(\mu')$  of the neighbour  $\mu'$  of  $\mu$  corresponding to  $\varepsilon$  when considering  $\mathcal{T}$  rooted at  $\mu$ .

The *SPQR-tree* of  $G$  with respect to the edge  $uv$ , originally introduced by Di Battista and Tamassia [8], is the (unique) smallest decomposition tree  $\mathcal{T}$  for  $G$ . Using a different edge  $u'v'$  of  $G$  and a composition of  $G - u'v'$  corresponds to rerooting  $\mathcal{T}$  at the node representing  $u'v'$ . It thus makes sense to say that  $\mathcal{T}$  is the SPQR-tree of  $G$ . The SPQR-tree of  $G$  has size linear in  $G$  and can be computed in linear time [11]. Planar embeddings of  $G$  correspond bijectively to planar embeddings of all skeletons of  $\mathcal{T}$ ; the choices are the orderings of the parallel edges in P-nodes and the embeddings of the R-node skeletons, which are unique up to a flip. When considering rooted SPQR-trees, we assume that the embedding of  $G$  is such that the root edge is incident to the outer face, which is equivalent to the parent edge being incident to the outer face in each skeleton. We remark that in a planar embedding of  $G$ , the poles of any node  $\mu$  of  $\mathcal{T}$  are incident to the outer face of  $\text{pert}(\mu)$ . Hence, in the following we only consider embeddings of the pertinent graphs with their poles lying on the same face and refer to such embeddings as *regular*.

Let  $\mu$  be a node of  $\mathcal{T}$ , we denote the poles of  $\mu$  by  $u(\mu)$  and  $v(\mu)$ , respectively. In the remainder of the paper, we will assume edge  $(u(\mu), v(\mu))$  to be part of  $\text{skel}(\mu)$  and  $\text{pert}(\mu)$ . The outer face of a (regular) embedding of  $\text{pert}(\mu)$  is the one obtained from such an embedding after removing the  $(u(\mu), v(\mu))$  connecting its poles. Also, the two paths incident to the outer face of  $\text{pert}(\mu)$  between  $u(\mu)$  and  $v(\mu)$  are called *boundary paths* of  $\text{pert}(\mu)$ .

### A.3 Canonical Ordering

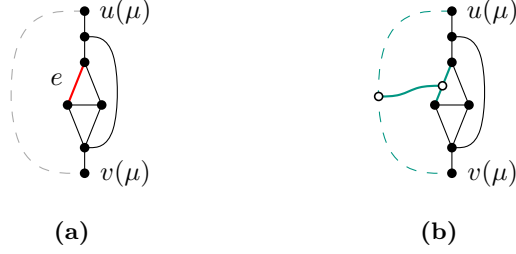
Let  $G = (V, E)$  be a 3-connected plane graph with vertices  $v_2, v_1$ , and  $v_n$  in this clockwise order along the outer face of  $G$ . Let  $\pi = (P_1, \dots, P_k)$  be an ordered partition of  $V$  into paths, where  $P_1 = (v_1, v_2)$  and  $P_k = (v_n)$ . Define  $G_i$  to be the subgraph of  $G$  induced by  $P_1 \cup \dots \cup P_i$ , and denote by  $C_i$  the boundary of the outer face of  $G_i$ . We say that  $\pi$  is a canonical ordering for  $G$  if:

- each  $C_i (i > 1)$  is a cycle containing  $(v_1, v_2)$ .
- each  $G_i$  is biconnected and internally 3-connected, that is, removing two interior vertices of  $G_i$  does not disconnect it; and
- for each  $i \in \{2, \dots, k-1\}$ , one of the two following conditions holds:
  - (a)  $P_i$  is a singleton,  $\{z\}$ , where  $z$  belongs to  $C_i$  and has at least one neighbor in  $G \setminus G_i$ .
  - (b)  $P_i$  is a chain,  $\{z_1, \dots, z_l\}$ , where each  $z_j$  has at least one neighbour in  $G \setminus G_i$ , and where  $z_1$  and  $z_l$  each have one neighbour on  $C_{i-1}$ , and these are the only two neighbors of  $P_i$  in  $G_{i-1}$ .

Observe that, by condition (b), if  $P_i$  is a chain, then the two neighbours of  $z_1$  and  $z_l$  on  $C_{i-1}$  are adjacent in  $G_{i-1}$ .

## B Bounded-degree Augmentation

In this section we give an algorithm (Lemma 1) to augment a 2-connected planar graph  $G$  with minimum degree  $\delta(G) \geq 3$  and maximum degree  $\Delta(G)$  to a 3-



**Fig. 6:** (a) Pertinent graph of an  $e$ -externally 3-connectible S-node  $\mu$  and (b) the subdivision of a 3-connected planar graph obtained by performing the  $H$ -split on  $\langle e, (u(\mu), v(\mu)) \rangle$  in graph  $\text{pert}(\mu)$ .

connected planar graph  $G'$  with  $\delta(G') = \delta(G)$  and  $\Delta(G') = \Delta(G)$  by applying  $H$ -splits to disjoint pairs of edges of  $G$ .

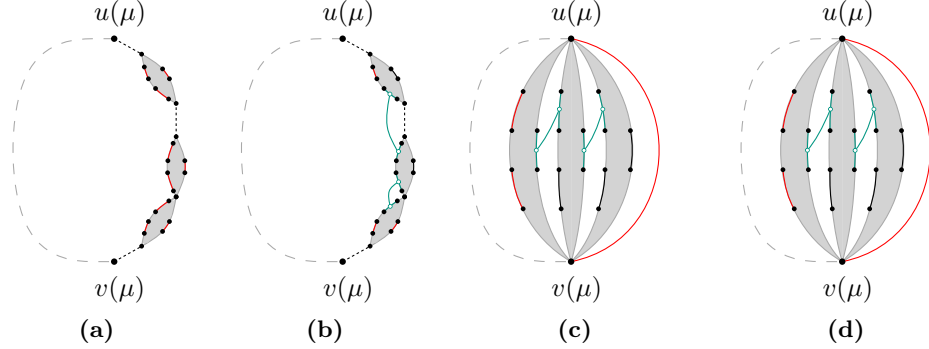
We start with some definitions. Let  $e$  be an edge of  $\text{pert}(\mu)$  incident the outer face of some regular embedding  $\mathcal{E}_\mu$  of  $\text{pert}(\mu)$ . Observe that, by the definition of outer face of a pertinent graph, it holds that  $e \neq (u(\mu), v(\mu))$ . Then,  $\mathcal{E}_\mu$  is  $e$ -externally 3-connectible if either 1.  $\mu$  is a Q-node, that is,  $\text{pert}(\mu) = e$ , or 2. the graph obtained from the  $\text{pert}(\mu)$  by performing an  $H$ -split on  $\langle e, (u(\mu), v(\mu)) \rangle$  is a subdivision of a 3-connected graph whose unique subdivision vertices are the poles of  $\mu$ ; refer to Fig. 6. Also, we say that  $\mu$  is  $e$ -externally 3-connectible (or, simply, *externally 3-connectible*) if  $\text{pert}(\mu)$  admits an  $e$ -externally 3-connectible embedding, for some edge  $e$  of  $\text{pert}(\mu)$ .

Let  $\mu$  be a node of  $\mathcal{T}$ , let  $e$  be an edge of  $\text{pert}(\mu)$ , and let  $\text{pert}^*(\mu)$  be a graph obtained by applying  $H$ -splits on distinct pairs of edges in  $\text{pert}(\mu)$ . We say that  $e$  is a *free edge* if  $e \in E(\text{pert}(\mu)) \setminus E(\text{pert}^*(\mu))$ , that is, edge  $e$  has not been used in any  $H$ -split.

Consider a non-Q-node  $\mu$ . Let  $\text{pert}^*(\mu)$  be a graph obtained from  $\text{pert}(\mu)$  via a set of edge-disjoint  $H$ -splits, let  $L_\mu = [e_1, e_2]$  and  $R_\mu = [e_3]$  be two lists of free edges in  $E(\text{pert}^*(\mu))$ , and let  $\mathcal{E}_\mu^*$  be a regular embedding of  $\text{pert}^*(\mu)$ . We say that the 4-tuple  $\langle \text{pert}^*(\mu), \mathcal{E}_\mu^*, L_\mu, R_\mu \rangle$  is *extendible* if  $L_\mu$  and  $R_\mu$  are incident to different faces of  $\mathcal{E}_\mu^*$  incident to the parent edge and  $\mathcal{E}_\mu^*$  is  $e$ -externally 3-connectible with  $e \in L_\mu$ . Observe that, once  $\text{pert}^*(\mu)$ ,  $L_\mu$  and  $R_\mu$  have been fixed, there exists a unique (up to a flip) embedding  $\mathcal{E}_\mu^*$  of  $\text{pert}^*(\mu)$  such that  $\langle \text{pert}^*(\mu), \mathcal{E}_\mu^*, L_\mu, R_\mu \rangle$  is extendible. Hence, to easy the notation, in the following we will omit to specify the embedding of  $\text{pert}^*(\mu)$ . If  $\mu$  is a Q-node representing edge  $e = (u(\mu), v(\mu))$ , then we also say that the triple  $\langle e, L_\mu = [e], R_\mu = [e] \rangle$  is *extendible*, thus allowing  $|L_\mu| = 1$  and  $L_\mu \cap R_\mu \neq \emptyset$  in this case.

For simplicity, we will also use the notation  $\text{exp}^*(e_\mu)$  to refer to  $\text{pert}^*(\mu)$  where  $e_\mu$  is the virtual edge representing  $\mu$  in the skeleton of its parent.

Let  $\mu$  be an internal node in  $\mathcal{T}$  with children  $\mu_1, \dots, \mu_k$  and let  $f$  be a face of an embedding  $\mathcal{E}_\mu$  of  $\text{skel}(\mu)$ . Consider the clockwise sequence of virtual



**Fig. 7:** Illustration for the proof of Lemma 1 when  $\mu$  is ((a),(b)) an S-node with three non-Q-node children  $\nu_1, \nu_2$ , and  $\nu_3$ , and ((c),(d)) a P-node with a Q-node child. ((a),(c)) Auxiliary graph  $\text{pert}(\mu)$ . ((b),(d)) Augmentation of  $\text{pert}(\mu)$  to  $\text{pert}^*(\mu)$  via  $H$ -splits.

edges in  $f$ . This sequence induces a natural order of the lists  $L_{\mu_i}$  of the children of  $\mu$  whose corresponding virtual edges bound  $f$  such that performing  $H$ -splits between consecutive pairs of free edges of such children does not violate planarity. Hence, in the following we will always assume lists  $L_{\mu_i}$  to be ordered according to such a natural order.

We root the SPQR-tree  $\mathcal{T}$  of  $G$  to an arbitrary Q-node  $\rho$  whose unique child  $\xi$  is an R-node. Observe that such a Q-node exists since  $\delta(G) \geq 3$ . We process the nodes of  $\mathcal{T}$  bottom-up and show how to compute for each node  $\mu$  with children  $\mu_1, \dots, \mu_k$  an extendible triple  $\langle \text{pert}^*(\mu), L_\mu, R_\mu \rangle$ , starting from the extendible triples  $\langle \text{pert}^*(\mu_i), L_{\mu_i}, R_{\mu_i} \rangle$  of its children. When we reach the root  $\rho$ , performing the  $H$ -split on  $\langle (u(\rho), v(\rho)), e \in L_\xi \rangle$  in the graph  $\text{pert}^*(\xi)$ , clearly yields a 3-connected planar graph with the desired properties, as the triple  $\langle \text{pert}^*(\xi), L_\xi, R_\xi \rangle$  is extendible.

We show how to compute extendible triples for each non-root node  $\mu \in \mathcal{T}$ .

Suppose  $\mu$  is a Q-node representing edge  $e = (u(\mu), v(\mu))$ . In this case there is nothing to be done as the triple  $\langle \text{pert}^*(\mu) = e, L_\mu = [e], R_\mu = [e] \rangle$  is extendible by definition.

Suppose  $\mu$  is an S-node with children  $\mu_1, \dots, \mu_k$ . Recall that since  $\mu$  is an internal node of  $\mathcal{T}$ , it has at least two children; also, since  $\delta(G) \geq 3$ , no two Q-nodes are adjacent in  $\text{skel}(\mu)$ . Hence,  $\mu$  has at least a non-Q-node child  $\nu$ . We distinguish two cases based on whether the number of non-Q-node children of  $\mu$  is larger than one or not.

**Case S1.** Assume  $\nu$  is the unique non-Q-node child of  $\mu$  and let  $e_\nu$  be the corresponding virtual edge in  $\text{skel}(\mu)$ . We construct an extendible triple for  $\mu$  as follows. Let  $\text{pert}(\mu)$  be the graph obtained from  $\text{skel}(\mu)$  by replacing  $e_\nu$  in  $\text{skel}(\mu)$  with  $\text{exp}^*(e_\nu)$ . We set  $\text{pert}(\mu^*) = \text{pert}(\mu)$ ,  $L_\mu = L_\nu$ , and  $R_\mu = R_\nu$ . The fact that the constructed triple is extendible is due to the fact that (i)  $\langle \text{pert}^*(\nu), L_\nu, R_\nu \rangle$

is extendible and that (ii) since  $\nu$  is the unique non-Q-node child of  $\mu$  and since the remaining (at most two) Q-node children of  $\mu$  are not adjacent in  $\text{skel}(\mu)$ , the poles of the corresponding virtual edges cannot be part of a separation pair of  $\text{pert}^*(\mu)$ .

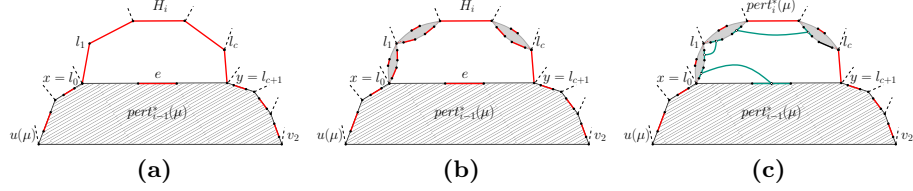
**Case S2.** Let  $\nu_1, \dots, \nu_s$  be the non-Q-node children of  $\mu$  ordered as the corresponding virtual edges appear in  $\text{skel}(\mu)$  from  $u(\mu)$  to  $v(\mu)$ . We construct an extendible triple for  $\mu$  as follows; refer to Fig. 7. First, we construct an auxiliary graph  $\text{pert}(\mu)$  starting from  $\text{skel}(\mu)$  by replacing each virtual edge  $e_{\nu_i}$  in  $\text{skel}(\mu)$  corresponding to a non-Q-node child  $\nu_i$  of  $\mu$  with the expansion graph  $\text{exp}^*(e_{\nu_i})$  of  $\nu_i$ , for  $i = 1, \dots, s$ ; see Fig. 7a. Then, we obtain  $\text{pert}^*(\mu)$  from  $\text{pert}(\mu)$  by performing an  $H$ -split on  $\langle L_{\nu_i}[2], L_{\nu_{i+1}}[1] \rangle$ , for  $i = 1, \dots, s-1$ ; see Fig. 7b. Finally, we set  $L_\mu = [L_{\nu_1}[1], L_{\nu_s}[2]]$  and  $R_\mu = R_{\nu_1}$ . We now show that the constructed triple is extendible. Observe that each  $H$ -split can be seen as an operation that turns two non-Q-node children of  $\mu$ , together with the unique Q-node possibly separating them in  $\text{skel}(\mu)$ , into a single externally 3-connectible child of  $\mu$ . Hence, at the end of the augmentation, there might exist at most two non-adjacent virtual edges representing Q-node children of  $\mu$  left in  $\text{skel}(\mu)$ , whose poles do not contribute to any separation pair of  $\text{pert}^*(\mu)$ .

Suppose  $\mu$  is an P-node. Recall that at most one child of  $\mu$  can be a Q-node. We construct an extendible triple for  $\mu$  as follows; refer to Fig. 7. First, we select an arbitrary embedding of  $\text{skel}(\mu)$  such that the unique child of  $\mu$  that is a Q-node, if any, is incident to the outer face. Let  $\nu_1, \dots, \nu_s$  be the clockwise ordering of the non-Q-node children of  $\mu$  around  $u(\mu)$  determined by such an embedding, where  $\nu_1$  is a non-Q-node child of  $\mu$  incident to the outer face. Second, we construct an auxiliary graph  $\text{pert}(\mu)$  starting from  $\text{skel}(\mu)$  by replacing each virtual edge  $e_{\nu_i}$  in  $\text{skel}(\mu)$  with  $\text{exp}^*(e_{\nu_i})$ , for  $i = 1, \dots, s$ ; see Fig. 7c. Third, we obtain  $\text{pert}^*(\mu)$  from  $\text{pert}(\mu)$  by performing an  $H$ -split on  $\langle R_{\nu_i}[1], L_{\nu_{i+1}}[2] \rangle$ , for  $i = 1, \dots, s-1$ ; see Fig. 7d. Finally, we set  $L_\mu = L_{\nu_1}$ , and  $R_\mu = R_{\nu_s}$ , if there exists no Q-node child of  $\mu$ , or  $R_\mu = [(u(\mu), v(\mu))]$ , otherwise.

Suppose  $\mu$  is an R-node. Let  $\mathcal{E}_\mu$  be the unique (up to a flip) regular embedding of  $\text{skel}(\mu)$  and let  $\pi = (P_1 = (u(\mu), v_2), P_2, \dots, P_k = v(\mu))$  be a canonical ordering of  $\text{skel}(\mu)$  where  $v_2$  is the neighbour of  $u(\mu)$  different from  $v(\mu)$  that is incident to the outer face of  $\mathcal{E}_\mu$ . Also, let  $\text{skel}_i(\mu)$  be the embedded subgraph of  $\text{skel}(\mu)$  induced by vertices  $\bigcup_{j=1}^i P_j$  and  $C_i$  be the cycle bounding the outer face of  $\text{skel}_i(\mu)$ . Further, let  $\text{pert}_i(\mu)$  be the graph obtained from  $\text{skel}_i(\mu)$  as illustrated before.

We show how to augment  $\text{pert}_i(\mu)$  to a new graph  $\text{pert}_i^*(\mu)$  via  $H$ -splits so that  $\text{pert}_i^*(\mu)$  is internally 3-connected and the outer face of  $\text{pert}_i^*(\mu)$  contains at least  $|C_i| - 1$  free edges, each of which is separated by two vertices of  $\text{skel}_i(\mu)$ . Hence,  $\text{pert}_k^*(\mu)$  contains free edges on its outer face that can be used to instantiate  $L_\mu$  and  $R_\mu$ . Further, since  $\text{skel}_k(\mu) = \text{skel}(\mu)$  is 3-connected, so is  $\text{pert}_k^*(\mu) = \text{pert}^*(\mu)$ . Hence,  $\text{pert}^*(\mu)$  is trivially  $e$ -externally 3-connectible with respect to any edge  $e \in L_\mu$ .

The augmentation is done by induction on  $i$ . We first assume that the child of  $\mu$  corresponding to edge  $\ell = (u(\mu), v_2)$  is a Q-node; we will show how to remove



**Fig. 8:** Illustration for the proof of Lemma 1 when  $\mu$  is an R-node. Graphs (a)  $H_i$  and (b)  $\overline{H}_i$ . (c) Augmentation of  $\overline{H}_i$  to  $\text{pert}_i^*(\mu)$  via  $H$ -splits.

such an assumption at the end of the construction by performing a special  $H$ -split.

The base case is  $i = 1$ . In this case,  $\text{skel}_i(\mu)$  consists of the single virtual edge  $\ell$ . We obtain  $\text{pert}_i^*(\mu)$  by simply replacing edge  $\ell$  with  $\text{exp}^*(\ell)$ . Observe that, the outer face of  $\text{pert}_i^*(\mu)$  contains at least a free edge (in fact, exactly one since the child of  $\mu$  corresponding to  $e$  is a Q-node). Also,  $\text{pert}_i^*(\mu)$  is internally 3-connected, since  $\text{exp}^*(\ell)$  is externally 3-connectible.

In the inductive step  $1 < i \leq k$ , assume we have already computed graph  $\text{pert}_{i-1}^*(\mu)$ . By the inductive hypothesis,  $\text{pert}_{i-1}^*(\mu)$  is internally 3-connected and the outer face of  $\text{pert}_{i-1}^*(\mu)$  contains at least  $|C_{i-1}| - 1$  free edges, each of which is separated by two vertices of  $\text{skel}_{i-1}(\mu)$ . Let  $P_i = (l_1, \dots, l_c)$  be the  $i$ -th element in  $\pi$ . Suppose  $|P_i| > 1$ , that is,  $P_i$  is not a singleton vertex; the case  $|P_i| = 1$  being simpler. Observe that, since  $\pi$  is a canonical ordering, there exists a virtual edge  $(l_1, l_k)$  that is incident to the outer face of  $\text{skel}_{i-1}(\mu)$ . Let  $e$  be the free edge along the outer face of  $\text{pert}_{i-1}^*(\mu)$  between  $l_1$  and  $l_k$  and belonging to the expansion graph of virtual edge  $(l_1, l_k)$ . We construct  $\text{pert}_i^*(\mu)$  as follows. First, initialise  $\text{pert}_i^*(\mu)$  to  $\text{pert}_{i-1}^*(\mu)$ . Let  $P_i^* = (x = l_0, l_1, \dots, l_c, l_{c+1} = y)$  be the list of vertices obtained by prepending and appending to  $P_i$  vertices  $x$  and  $y$ , respectively, where  $x$  and  $y$  are the only two neighbours of  $l_1$  and  $l_c$  in  $\text{pert}_i^*(\mu)$ , respectively. Add to  $\text{pert}_i^*(\mu)$  vertices  $l_1, \dots, l_c$  in the outer face of  $\text{pert}_i^*(\mu)$  and edges  $(l_j, l_{j+1})$ , with  $0 \leq j < c$ . Denote by  $\nu_j$  the child of  $\mu$  corresponding to the virtual edge  $e_j$ , for  $j = 1, \dots, c$ , and let  $H_i$  be the resulting graph; see Fig. 8a. Then, replace each edge  $e_j = (l_j, l_{j+1})$  with  $\text{exp}^*(e_j)$ , for  $j = 0, \dots, c$  (except, when  $i = k$ , for the edge representing the parent of  $\mu$ ). Let  $\overline{H}_i$  be the resulting graph; see Fig. 8b. Finally, we obtain  $\text{pert}_i^*(\mu)$  from  $\overline{H}_i$  by performing an  $H$ -split (i) on  $\langle e, L_{\nu_1}[1], \rangle$  and (ii) on  $\langle L_{\nu_j}[2], L_{\nu_{j+1}}[1] \rangle$ , for  $j = 1, \dots, c - 1$ ; see Fig. 8c. Graph  $\text{pert}_i^*(\mu)$  is internally 3-connected, since  $\text{skel}_i(\mu)$  is internally 3-connected and since once an  $H$ -split is performed the poles of the virtual edge  $e_{\nu_j}$  of  $\text{skel}_i(\mu)$  whose expansion graph  $\text{exp}^*(e_{\nu_j})$  interested by the  $H$ -split do not belong to a separation pair in  $\text{pert}_i^*(\mu)$ . Further, there exists a free edge on the outer face of  $\text{pert}_i^*(\mu)$  between each pair of consecutive vertices in  $(l_0, l_1, \dots, l_c, l_{c+1})$ , namely, for each virtual edge  $e_j = (l_j, l_{j+1})$  with  $0 \leq j \leq c$ , edge  $R(\nu_j)$  is incident to the outer face of  $\text{pert}_i^*(\mu)$ .

To complete the proof, we only need to show that the child of  $\mu$  corresponding to the virtual edge  $\ell = (u(\mu), v_2)$  needs not to be a Q-node. Observe that, in the inductive construction we didn't make use of the free edge corresponding to edge  $\ell$  in any  $H$ -split. Let  $\nu_\ell$  be the child of  $\mu$  corresponding to  $\ell$  and let  $\tau$  be the child of  $\mu$  whose corresponding virtual edge is incident to  $\ell$  and to the outer face of  $\text{skel}(\mu)$ . Once  $\text{pert}_k^*(\mu)$  has been constructed, we replace  $\ell$  with  $\text{exp}^*(\ell)$  and perform an  $H$ -split on  $\langle L_{\nu_\ell}[2], L_\tau[1] \rangle$ .

Altogether we have proved the following main result.

**Lemma 1.** *Let  $G = (V, E)$  be a 2-connected planar graph with minimum degree  $\delta(G) \geq 3$  and maximum degree  $\Delta(G)$ . There exist disjoint pairs  $(e'_1, e''_1), \dots, (e'_k, e''_k)$  of edges in  $E$  such that performing the  $H$ -splits  $\langle e'_1, e''_1 \rangle, \dots, \langle e'_k, e''_k \rangle$  yields a 3-connected planar graph  $G'$  with  $\delta(G') = \delta(G)$  and  $\Delta(G') = \Delta(G)$ .*

## C Omitted Proofs from Section 4

**Theorem 1.** *MIS is NP-complete for 3-connected cubic planar graphs.*

*Proof.* Let  $G$  be a 2-connected cubic planar graph. By Lemma 1, a 3-connected cubic planar graph  $G'$  can be obtained from  $G$  by applying  $H$ -splits on  $k \in O(n)$  distinct pairs of edges in  $E(G)$ .

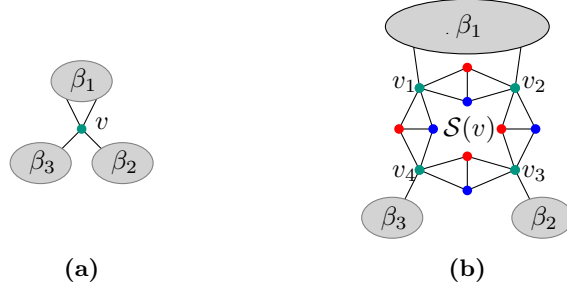
Let  $\langle (a_1, b_1), \dots, (a_k, b_k) \rangle$  be any ordering of the set of edge pairs of  $E(G)$  determined by the algorithm described in the proof of Lemma 1. We augment graph  $G$  to an auxiliary graph  $G^*$  as follows. For  $i = 1, \dots, k$ , we apply the construction illustrated in Fig. 3a to the edge pair  $(a_i, b_i)$ . It is easy to see that  $G^*$  is planar, cubic, and 3-connected. Furthermore, graph  $G^*$  can be inductively defined as follows. Let  $G_0 = G$  and  $G_i$  be the graph obtained by applying the construction illustrated in Fig. 3a to the edge pair  $(a_i, b_i)$  of graph  $G_{i-1}$ . Then, we have that  $G^* = G_k$ .

By Lemma 2, it holds that  $\alpha(G_i) = \alpha(G_{i-1}) + 5$ , with  $i = 1, \dots, k$ . It follows that  $G$  admits an independent set of size  $w$  if and only if  $G^* = G_k$  admits an independent set of size  $w + 5k$ . Also,  $|V(G^*)| = |V(G)| + 12k$ . Since  $k \in O(n)$  and since each graph  $G_i$  can be obtained by  $G_{i-1}$  in constant time, this is a polynomial-time reduction from MIS in 2-connected cubic planar graphs to MIS in 3-connected cubic planar graphs. The fact that the former is NP-complete [12] implies the claim.  $\square$

**Lemma 3.** *The 3-COLORING problem is NP-complete for 2-connected planar graphs with minimum degree 4 and maximum degree 7.*

*Proof.* We show a reduction from the 3-COLORING problem for 4-regular planar graphs [7].

Let  $G$  be a 4-regular plane graph  $G$ . First, we replace each cut vertex  $v$  incident to more than two blocks with the gadget  $\mathcal{S}(v)$  illustrated in Fig. 9b to obtain a new plane graph  $G'$ . Observe that,  $G'$  is 3-colorable if and only if  $G$  is and that each cut vertex of  $G'$  is now incident to exactly two blocks. Further, it is easy to verify that  $\Delta(G') \leq 5$ .



**Fig. 9:** (a) A cut vertex  $v$  incident to three blocks  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$ . (b) Gadget  $S(v)$  that replaces  $v$  with new cut vertices each incident to exactly two blocks.

Second, let  $(v, x)$  and  $(v, y)$  be two edges of  $G'$  appearing consecutively around a cut vertex  $v$  and each belonging to a different block. We augment  $G'$  to a minimum degree 4 and maximum degree 6 planar graph  $G''$  such that  $G'$  is 3-colorable if and only if  $G''$  is and  $\mathcal{C}(G'') = \mathcal{C}(G') - 1$  by replacing the pair  $\langle (v, x), (v, y) \rangle$  with  $A_1(\langle (v, x), (v, y) \rangle)$  (see Fig. 3b). Repeating such an augmentation for each cut vertex, eventually yields a 2-connected planar graph  $G^*$  that is 3-colorable if and only if  $G$  is. Further, since each augmentation increases the degree of the end points of the selected edge pair by at most 1 and since each edge might be involved in at most two augmentations (if both its end points are cut vertices), it follows that  $\Delta(G^*) \leq \Delta(G') + 2 = 7$ .  $\square$

## D Omitted Proofs from Section 5

**Lemma 4.** *Let  $G$  be a 2-connected plane graph and  $f_{\geq 4}(G)$  be the number of faces of  $G$  whose size is larger than 3. There exists a 2-connected plane graph  $G'$  such that (i)  $\alpha(G') = \alpha(G) + 1$  and (ii)  $f_{\geq 4}(G') = f_{\geq 4}(G) - 1$ .*

*Proof.* Let  $f$  be any non-triangular face of  $G$  and let  $\ell(f)$  be the length of  $f$ . Also, for any  $k \geq 4$ , let  $\Phi_k$  be the graph constructed as follows. First, initialize  $\Phi_k$  to the union of a cycle  $C_k = c_1, c_2, \dots, c_k$  of length  $k$  and of the complete graph on the four vertices  $\{a, b, c, d\}$ . Then, add to  $\Phi_k$  edges  $(d, c_i)$ , for  $i = 1, 2, \dots, k$ , and edges  $(c_1, d)$ ,  $(c_1, b)$ ,  $(c_1, c)$  and  $(c_2, c)$ . Observe that,  $\Phi_k$  is internally triangulated and hence 2-connected. Refer to Fig. 4 for an illustration of the gadget  $\Phi_6$ . Graph  $G'$  can be obtained from  $G$  by identifying  $f$  with the outer face of gadget  $\Phi_{\ell(f)}$ . Clearly, graph  $G'$  is 2-connected as  $G$  and  $\Phi_{\ell(f)}$  are. Also, since face  $f$  has been replaced by an internally triangulated graph, it is  $f_{\geq 4}(G') = f_{\geq 4}(G) - 1$ .

We now prove that  $\alpha(G') = \alpha(G) + 1$ . First, observe that an independent set  $I'$  in  $G'$  can contain at most one vertex in  $\{a, b, c, d\}$ , since these vertices induce a  $K_4$ . Hence,  $I' \cap V(G)$  is an independent set of size  $\alpha(G') - 1$ , showing that  $\alpha(G') \leq \alpha(G) + 1$ . Conversely, if  $I$  is an independent set in  $G$ , then  $I' \cup \{a\}$  is an independent set in  $G'$ , showing that  $\alpha(G') \geq \alpha(G) + 1$ .  $\square$



**Theorem 3.** MIS is NP-complete for planar triangulations.

*Proof.* Let  $G$  be a 2-connected plane graph. Observe that applying the reduction from Lemma 4 to a face of  $G$  whose size is larger than 3 yields a 2-connected plane graph  $G'$  where  $\alpha(G') = \alpha(G) + 1$  and the number of non-triangular faces of  $G'$  is one less the number of non-triangular faces of  $G$ . Iterating this reduction eventually leads to a planar triangulation  $G^*$  with  $\alpha(G^*) = \alpha(G) + f_{\geq 4}(G)$ , where  $f_{\geq 4}(G)$  denotes the number of faces of size at least four in  $G$ . Hence,  $G$  admits an independent set of size  $k$  if and only if  $G^*$  admits an independent set of size  $k + f_{\geq 4}(G)$ .

It follows that the described procedure is a polynomial-time reduction from MIS in 2-connected planar graphs to MIS in planar triangulations. The fact that the former is NP-complete [12] implies the claim.  $\square$

**Lemma 5.** STEINER TREE is NP-complete for biconnected planar graphs of maximum degree 3.

*Proof.* We start with an instance  $(G, T)$  of (unweighted) STEINER TREE, where  $G$  is a biconnected planar graph with  $n$  vertices and  $m$  edges, and  $T \subseteq V(G)$  is a set of terminals. This problem is known to be NP-complete [10]. We first construct a new instance  $(G', T')$  of WEIGHTED STEINER TREE where  $G'$  is biconnected and has maximum degree-3 by replacing each vertex  $v$  of degree more than 3 by a cycle  $C_v$  of the same length so that  $G'$  is planar. The edges of the cycles have weight 1, and we give the remaining edges a weight of  $2m + 1$ . For each  $v \in V(G)$ , if  $v$  has degree at most 3, it is  $v \in T'$ , otherwise, we choose an arbitrary vertex  $u \in C_T$  and put it in  $T'$ .

A Steiner tree with  $k$  edges in  $G$  can be augmented to a Steiner tree in  $G'$  of weight  $w$  with  $(2m + 1)k < w < (2m + 1)k + 2m$  by adding for cycle  $C_T$  all except one of the edges. Conversely, a Steiner tree in  $G'$  of weight  $w$  yields a Steiner tree in  $G$  with  $\lfloor w/(2m + 1) \rfloor$  edges. The reduction can be performed in polynomial time.

Finally, to get rid of the weights, we subdivide each edge of weight  $w$  by  $w - 1$  subdivision vertices. This shows that also the unweighted version is hard for these graphs. Observe that this is a polynomial-time reduction since the weights above are polynomially bounded in the input size.  $\square$